

# Mittag-Leffler functions as generalizations of the exponential

Induced groups and related transforms

Gorka Guardiola Múzquiz

GSYC

February 12, 2024



This document is shared with license Attribution-ShareAlike 4.0 International [1] (cc-by-sa).

## You are free to

- **ShareAlike** — *Copy and redistribute the material in any medium or format for any purpose, even commercially.*
- **Adapt** — *Remix, transform, and build upon the material for any purpose, even commercially.*
- **Some of this restrictions can be dispensed with permission from the authors.**
- *Third party images keep their original license.*

Under the following terms:

- **Attribution** — *You must give appropriate credit, provide a link to the license, and indicate if changes were made. You may do so in any reasonable manner, but not in any way that suggests the licensor endorses you or your use.*
- **ShareAlike** — *If you remix, transform, or build upon the material, you must distribute your contributions under the same license as the original.*

©2024 Gorka Guardiola Múzquiz.

[1] Some rights reserved. "Attribution-ShareAlike 4.0 International" (cc-by-sa). To view a copy of this license, visit <https://creativecommons.org/licenses/by-sa/4.0/deed.es>. or send a letter to Creative Commons, 559 Nathan Abbott Way, Stanford, California 94305, USA.

# What the talk is about

- Results from [1]: using Mittag-Leffler functions as generalizations of the exponentials in several forms.
- An introduction in layman's terms can be found in:  
<https://paureahack.blogspot.com/2016/06/supersum-subproduct.html>
- Questions I have for future work.

---

[1]Gorka Guardiola Múzquiz. “Mittag–Leffler functions, lpn-balls and geometric pairings”. In: *Integral Transforms and Special Functions* 28.3 (2017), pp. 195–211. DOI: 10.1080/10652469.2016.1267730.

# Outline

- 1 The question
- 2 Characterize the exponential
- 3 LCA Groups
- 4 Transforms
- 5 More groups
- 6 Time scales or measure chains
- 7 Summary
- 8 Future work

# Tetration (defined by iterated composition)

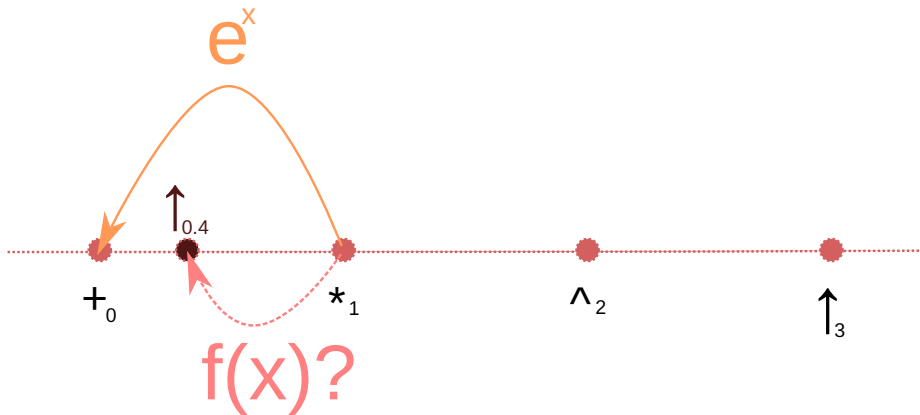
- Multiplication  $ab = a \uparrow_1 b = \underbrace{a + a \cdots a}_{b \text{ times}}$ .
- Exponentiation  $a^b = a \uparrow_2 b = \underbrace{aa \cdots a}_{b \text{ times}}$
- Tetration  $a \uparrow_3 b = \underbrace{a \left( a \left( a \cdots a \right) \right)}_{b \text{ times}}$ .
- N-ation (hyperoperations)  $a \uparrow_n b, N \in \mathbb{N} = \{1, 2, 3, 4, \dots\}$ .
- We can add 0 for addition and extend it.
- Addition  $a + b = a \uparrow_0 b = \underbrace{1 + 1 \cdots 1}_{b \text{ times}}$ .
- We can add negative numbers (superlogarithm, etc.) extend it to  $N \in \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ .
- Is it possible to extend it to  $\mathbb{R}$ ? What does  $a \uparrow_{0.5} b$  even mean?

# Fruitless idea

- What does  $a \uparrow_{0.5} b$  even mean?
- $f(x, y) = a + b$ ,  $ab = \underbrace{f(a, f(a, \dots f(a, a)))}_{b \text{ times}}$
- How do you multiply by 0.5? Use the inverse  $x * \frac{1}{2}$ , works for  $\mathbb{Q}$ , go from there.
- There is an implicit half-composition.
- Half-composition (like  $a^{0.5}$ /roots of matrices  $M^{\frac{1}{2}}$ /Chebyshev polynomials  $T_{\frac{n}{2}}(x)$ ).
- There may be something to this (see later).
- Did not work for me as a direct approach but...

# Shifting the exponential

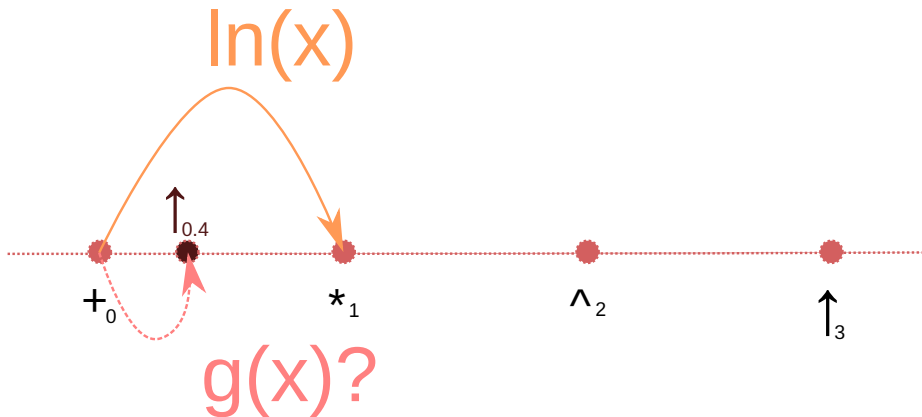
- $x, y \in \mathbb{R}$
- Take inspiration from Lie algebras (exponential map).
- $e^x e^y = e^{x+y}$ : homomorphism from multiplication to addition.
- What if we “shift” it?



- We need to find  $f(x)$  and characterize the resulting group.

# Shifting the logarithm

- $x, y \in \mathbb{R}^+$
- $\ln(x) + \ln(y) = \ln(xy)$ : homomorphism from addition to multiplication.
- What if we “shift” it?



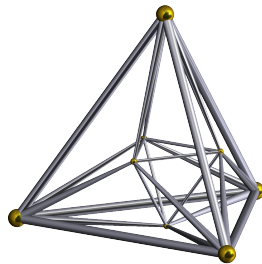
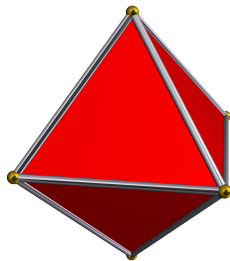
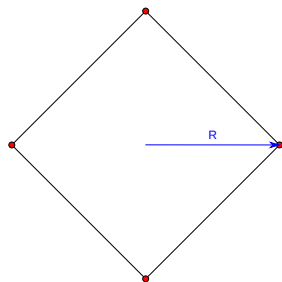
- We need to find  $g(x)$  and characterize the resulting group.

# Outline

- 1 The question
- 2 Characterize the exponential
- 3 LCA Groups
- 4 Transforms
- 5 More groups
- 6 Time scales or measure chains
- 7 Summary
- 8 Future work

# Properties of the exponential

- Completely monotonic (invertible).
- Holomorphic.
- Eigenfunction of the derivative.
- Generator function of volumes of hyperballs of  $I_1$  (cross-polytopes or hyper-octahedrons) as functions of outer radius.
- $e^z = \sum_0^\infty \frac{z^n}{n!}$
- $e^{2R} = \sum_0^\infty \frac{(2R)^n}{n!}$



# Mittag-Leffler functions

- Good references are [2] and [3] [4].
- One parameter:  $E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\alpha n)}$ ,  $\alpha \in \mathbb{R}, z \in \mathbb{C}$ .
- Two parameters (Wiman function):  $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta+\alpha k)}$ ,  $\alpha, \beta \in \mathbb{R}, z \in \mathbb{C}$ .
- We work mainly with one and two parameter Mittag-Leffler functions.

- 
- [2]H. J. Haubold, A. M. Mathai, and R. K. Saxena. “Mittag-Leffler functions and their applications”. In: *J. Appl. Math.* 2011 (2011), pp. 1–51. DOI: [10.1155/2011/298628](https://doi.org/10.1155/2011/298628).
- [3]A. Erdélyi et al. (*Based on notes left by Harry Bateman*) *Higher transcendental functions*. Vol. III. Melbourne: Robert E. Krieger Publishing Co. Inc., 1981.
- [4]Rudolf Gorenflo et al. *Mittag-Leffler functions, related topics and applications*. Springer, 2020.

# Mittag-Leffler functions (generalization)

- Further:  $E_{\alpha,\beta}^{\delta}(z) = \sum_{k=0}^{\infty} \frac{(\delta)_k z^k}{\Gamma(\beta + \alpha k)}$ ,  $\alpha, \beta, \delta \in \mathbb{R}, z \in \mathbb{C}$ , where  $(\gamma)_n$  is the generalized Pochhammer symbol,

$$(\gamma)_0 = 1,$$

$$(\gamma)_n = \gamma(\gamma + 1)(\gamma + 1) \cdots (\gamma + n - 1), \quad n \geq 1,$$

- Related also, generalized  $\gamma$ -hyperbolic functions:

$$F_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{\gamma^k z^{\beta + \alpha k}}{\Gamma(\beta + \alpha k + 1)} = z^{\beta} E_{\alpha,\beta+1}(\gamma z^{\alpha}), \quad \alpha, \beta \in \mathbb{R}, \quad \gamma, z \in \mathbb{C}.$$

- The Mittag-Leffler functions are a subfamily of the hypergeometric functions [5]:

$$E_{n,\beta}^{\gamma}(z) = \frac{1}{\Gamma(\beta)} {}_1F_n \left( \begin{matrix} \gamma \\ \beta/n, (\beta + 1)/n, \dots, (\beta + n - 1)/n \end{matrix}; \frac{z}{n^n} \right).$$

- Also related to Fox  $H$ -functions, Wright functions, Meijer  $G$ -functions, Rabotnov functions.

# Mittag-Leffler functions of one parameter

- $E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\alpha n)}, \alpha \in \mathbb{R}, z \in \mathbb{C}.$

# Mittag-Leffler functions of one parameter

- Completely monotonic (invertible),  $0 < \alpha \leq 1$  Pollard [5].
- Holomorphic.

---

[5]H. Pollard. “The completely monotonic character of the Mittag-Leffler function”. In: *Bull. Amer. Math. Soc.* 54.12 (1948), pp. 1115–1116.

# Mittag-Leffler functions of one parameter

- Eigenfunction of the fractional derivative (or iterated derivatives).
- $D^k E_k(\gamma z^k) = \gamma E_k(\gamma z^k)$ ,  $k \in \mathbb{N}$ ,  $\gamma \in \mathbb{R}$ .
- Given the right-sided operator of Riemann-Liouville fractional derivative,

$$(D_{0+}^\alpha f)(x) = \left( \frac{d}{dx} \right)^{[\alpha]+1} \left( I_{0+}^{1-\{\alpha\}} f \right)(x), \alpha > 0,$$

$$(I_{0+}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt.$$

► Where  $[\alpha]$  means the maximal integer number not exceeding  $\alpha$ ,  $\{\alpha\}$  is the fractional part of  $\alpha$ .

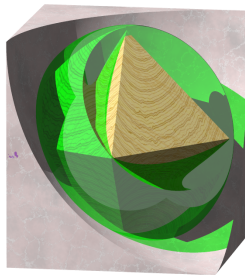
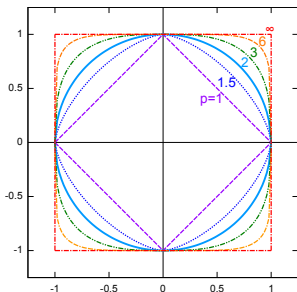
- From [6],  $\left( D_{0+}^\alpha [E_\alpha(\gamma t^\alpha)] \right)(z) = \frac{z^{-\alpha}}{\Gamma(1-\alpha)} + \gamma E_\alpha(\gamma z^\alpha)$ ,  $\alpha > 0$ ,  $a \in \mathbb{R}$ .
- Other definitions of fractional derivative can make the extra term disappear.

---

[6] R. K. Saxena and M. Saigo. "Certain Properties of Fractional Calculus Operators Associated with Generalized Mittag-Leffler Function". In: *Fract. Calc. Appl. Anal.* 8.2 (2005), pp. 141–154. ISSN: 1311-0454.

# Mittag-Leffler functions of one parameter

- Generator function of volumes of  $l_p$  hyperballs given the outer radius of with  $p \geq 1$ ,  $p = \frac{1}{\alpha}$ .
- Volume of  $B_n^p$  is  $V_n^p(R) = \frac{(2\Gamma(\frac{1}{p}+1)R)^n}{\Gamma(\frac{n}{p}+1)}$ . [7]
- $E_\alpha(2\Gamma(\alpha+1)R) = \sum_{n=0}^{\infty} \frac{(2\Gamma(\alpha+1)R)^n}{\Gamma(\alpha n+1)} = \sum_{n=0}^{\infty} V_n^{\frac{1}{\alpha}}(R)$ .



# Mittag-Leffler functions of one parameter

Some relevant special cases [8]:

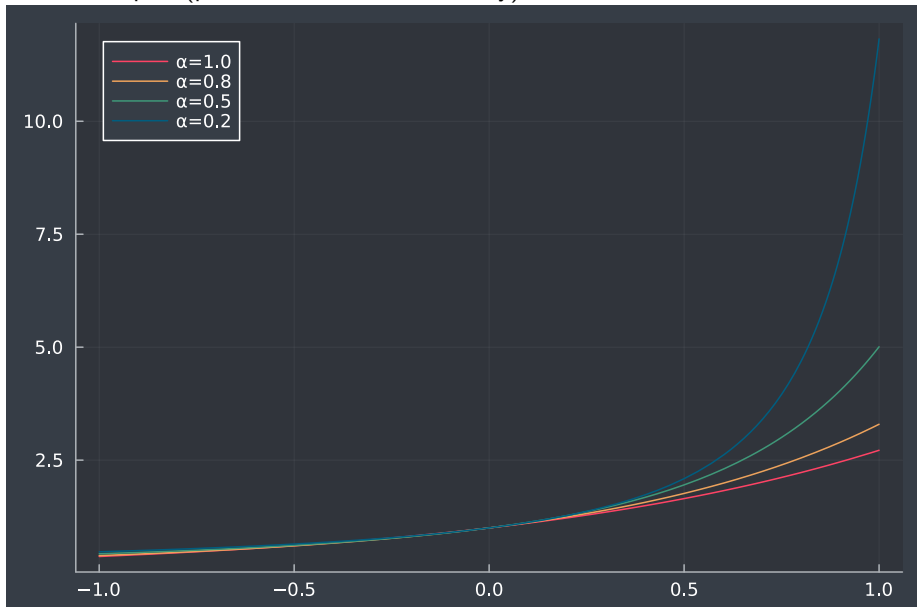
- $E_0(z) = \frac{1}{1-z}$ ,  $|z| < 1$ ,  $z \in \mathbb{C}$ , (hypercubes)
- $E_1(z) = e^z$ ,  $z \in \mathbb{C}$ , (cross-polytopes)
- $E_{\frac{1}{2}}(z) = e^{z^2}(1 - \operatorname{erf}(z)) = e^{z^2} \operatorname{erfc}(z)$ ,  $z \in \mathbb{C}$ , (hyperspheres)
  - ▶ With  $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2) dt$ .
- $E_2(z) = \cosh(\sqrt{z})$ ,  $z \in \mathbb{C}$ .

---

[8]H. J. Haubold, A. M. Mathai, and R. K. Saxena. “Mittag-Leffler functions and their applications”. In: *J. Appl. Math.* 2011 (2011), pp. 1–51. DOI: [10.1155/2011/298628](https://doi.org/10.1155/2011/298628).

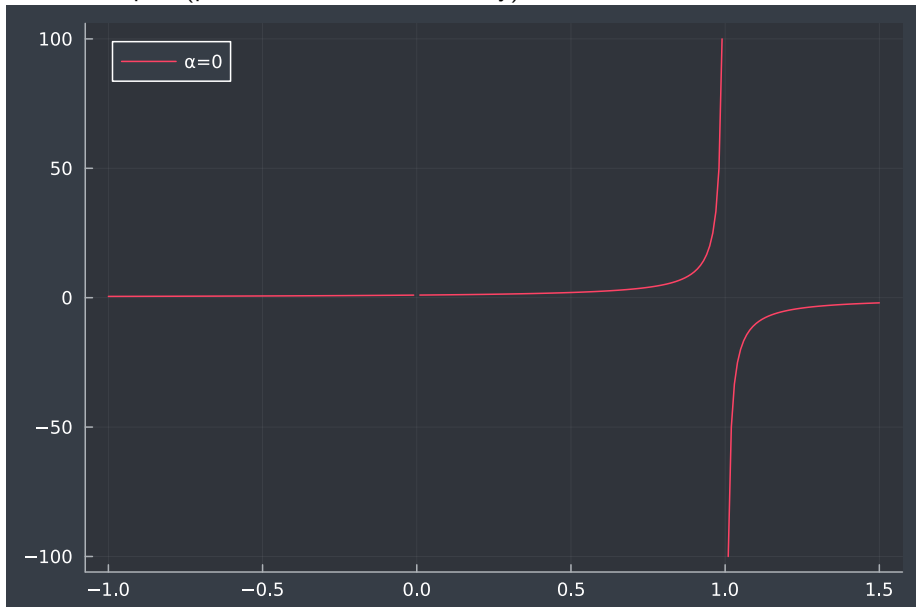
# Mittag-Leffler functions of one parameter

Some examples (plotted with the Julia library):



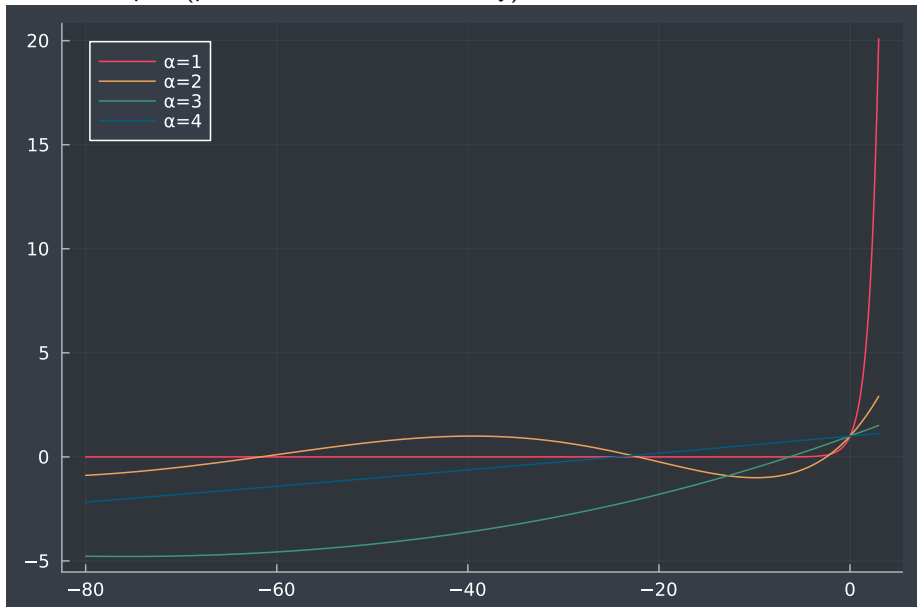
# Mittag-Leffler functions of one parameter

Some examples (plotted with the Julia library):



# Mittag-Leffler functions of one parameter

Some examples (plotted with the Julia library):



# Mittag-Leffler functions of two parameters (Wiman function)

- $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)}, \alpha, \beta \in \mathbb{R}, z \in \mathbb{C}.$
- Note that  $E_{\alpha,1}(z) = E_{\alpha}(z)$
- Completely monotonic (invertible) when  $\beta \geq \alpha, 0 < \alpha \leq 1$ . See Miller: [9].
- Holomorphic.

---

[9]K. S. Miller. “A note on the complete monotonicity of the generalized Mittag-Leffler function”. In: *Real Analysis Exchange* 23.2 (1999), pp. 753–756.

# Mittag-Leffler functions of two parameters (Wiman function)

- Generalizes the derivative properties.
- $\left( D_{0+}^{\alpha} [z^{\beta} E_{\alpha, \beta+1}(\gamma z^{\alpha})] \right) (z) = \frac{z^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)} + \gamma z^{\beta} E_{\alpha, \beta+1}(\gamma z^{\alpha}), \alpha > 0, \beta > 0.$

# Mittag-Leffler functions of two parameters (Wiman function)

- Generator function of generalized  $n$ -pseudocylinders,  $n$ -balls which have as sections in  $r$ -dimensional balls of  $\frac{1}{p} = \alpha$  and in  $q$  dimensions  $\frac{1}{s} = \beta$ .
- The volume of a  $n$ -pseudocylinders is:  $W_n^{p,s}(R) = \frac{\Gamma(\frac{1}{p}+1)^r \Gamma(\frac{1}{s}+1)^q (2R)^n}{\Gamma(\frac{r}{p} + \frac{s}{q} + 1)}$
- $\sum_{r=0}^{\infty} W_n^{p,s}(R) = (\Gamma(\beta + 1)2R)^q E_{\alpha, 1+q\beta}(\Gamma(\alpha + 1)2R), \alpha > 0, \beta \geq 0.$

# Mittag-Leffler functions of two parameters (Wiman function)

Some relevant special cases [10]:

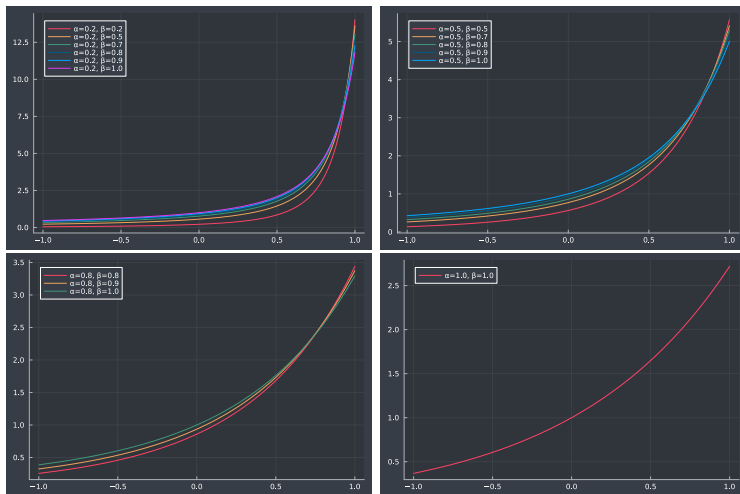
- $E_{1,2}(z) = \frac{e^z - 1}{z}, z \in \mathbb{C}$
- $E_{2,2}(z) = \frac{\sinh(\sqrt{z})}{\sqrt{z}}, z \in \mathbb{C}$
- $z^2 E_{\alpha, \beta+2\alpha}(z) = E_{\alpha, \beta}(z) - \frac{1}{\Gamma(\beta)} - \frac{z}{\Gamma(\alpha+\beta)}, z \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0$
- $E_{\frac{1}{2}, 2}(z) = \frac{e^{z^2} \operatorname{erfc}(z) - 1 - \frac{2z}{\sqrt{\pi}}}{z^2}, z \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0$

---

[10] H. J. Haubold, A. M. Mathai, and R. K. Saxena. "Mittag-Leffler functions and their applications". In: *J. Appl. Math.* 2011 (2011), pp. 1–51. DOI: [10.1155/2011/298628](https://doi.org/10.1155/2011/298628).

# Mittag-Leffler functions of two parameters (Wiman function)

- Some examples (plotted with the Julia library)



# Argument of the Mittag-Leffler functions

- $E_{\alpha,\beta}(x^\alpha)$  vs.  $E_{\alpha,\beta}(x)$ .
- When  $\alpha = 0$ ,  $x^\alpha$  not useful.
- $x^\alpha$  cannot be defined for  $\mathbb{R}$  when  $x < 0$ .
- Also, the volumes are in terms of  $E_{\alpha,\beta}(x)$ , an not  $E_{\alpha,\beta}(x^\alpha)$ .
- When  $\alpha > 1$ , or integer it may make sense, for example  $E_{2,\beta}(x^2)$ .

# Outline

- 1 The question
- 2 Characterize the exponential
- 3 LCA Groups**
- 4 Transforms
- 5 More groups
- 6 Time scales or measure chains
- 7 Summary
- 8 Future work

# Mittag-Leffler LCA groups

- $\beta \geq \alpha$ ,  $0 \leq \alpha \leq 1$
- Analogous to  $\ln(e^x e^y) = x + y$ ,  $x \in \mathbb{R}$ ,  $y \in \mathbb{R}$ :

$$z = x \circledast_{\alpha, \beta} y = E_{\alpha, \beta}^{-1}(E_{\alpha, \beta}(x)E_{\alpha, \beta}(y)). \quad (1)$$

- Analogous to  $e^{\ln(x)+\ln(y)} = xy$ ,  $x > 0$ ,  $y > 0$ :

$$z = x \oplus_{\alpha, \beta} y = E_{\alpha, \beta}(E_{\alpha, \beta}^{-1}(x) + E_{\alpha, \beta}^{-1}(y)). \quad (2)$$

# Mittag-Leffler LCA groups

- $z = x \circledast_{\alpha,\beta} y$ :
  - ▶ Commutativity, associativity: from multiplication.
  - ▶ Identity  $E_{\alpha,\beta}^{-1}(1)$ .
  - ▶ Inverse,  $\text{inv}_{\alpha,\beta}(x) = E_{\alpha,\beta}^{-1}\left(\frac{1}{E_{\alpha,\beta}(x)}\right)$ , exists by continuity.
- $z = x \oplus_{\alpha,\beta} y$ :
  - ▶ Commutativity, associativity: from addition.
  - ▶ Identity,  $E_{\alpha,\beta}(0) = \frac{1}{\Gamma(\beta)}$ .
  - ▶ Inverse,  $\text{inv}_{\alpha,\beta}(x) = E_{\alpha,\beta}(-E_{\alpha,\beta}^{-1}(x))$ , exists by continuity.
- $E_{\alpha,\beta}$  is a group homomorphism from  $*$  to  $\circledast$ :

$$E_{\alpha,\beta}(x \circledast_{\alpha,\beta} y) = E_{\alpha,\beta}(E_{\alpha,\beta}^{-1}(E_{\alpha,\beta}(x)E_{\alpha,\beta}(y))) = E_{\alpha,\beta}(x)E_{\alpha,\beta}(y).$$

- $E_{\alpha,\beta}^{-1}$  is a group homomorphism from  $+$  to  $\oplus$ .

$$E_{\alpha,\beta}^{-1}(x \oplus_{\alpha,\beta} y) = E_{\alpha,\beta}^{-1}(E_{\alpha,\beta}(E_{\alpha,\beta}^{-1}(x) + E_{\alpha,\beta}^{-1}(y))) = E_{\alpha,\beta}^{-1}(x) + E_{\alpha,\beta}^{-1}(y).$$

# Mittag-Leffler LCA groups

- $z = x \circledast_{\alpha,\beta} y$ :
  - ▶  $z = x \circledast_{1,1} y = x + y$ . (regular addition)
  - ▶  $z = x \circledast_{0,1} y = x + y - xy$  (a kind of multiplication, shifted/negated).
- In general, using  $E_{\alpha,\beta}(-x)$  tends to be simpler.
  - ▶  $z = x \circledast^{\alpha,\beta} y = -E_{\alpha,\beta}^{-1}(E_{\alpha,\beta}(-x)E_{\alpha,\beta}(-y))$ .
  - ▶ Then  $z = x \circledast^{0,1} = x + y + xy$ .
  - ▶  $z + 1 = (x + 1)(y + 1)$ : easier to interpret as multiplication.
  - ▶ It is also the OR operation (if  $+$  is XOR and multiplication is AND).
- **We can vary  $\alpha$  in  $\circledast_{\alpha,1}$  from multiplication to addition!!**
- The discontinuity: the absorbing element.

# Mittag-Leffler LCA groups: iterated composition

- From  $\circledast_{\alpha,\beta}$ :

- $$\underbrace{x \circledast_{\alpha,\beta} x \circledast_{\alpha,\beta} \cdots \circledast_{\alpha,\beta} x}_n = E_{\alpha,\beta}^{-1}(E_{\alpha,\beta}(x)^n) = x \uparrow_{\alpha,\beta} n,$$
- Generalized to  $x \uparrow_{\alpha,\beta} y = E_{\alpha,\beta}^{-1}(E_{\alpha,\beta}(x)^y)$
- Behaves like powers  $(x \uparrow_{\alpha,\beta} y) \uparrow_{\alpha,\beta} z = x \uparrow_{\alpha,\beta} yz$
- $x \uparrow_{\alpha,\beta} -1$  is the inverse of  $x$  w.r.t.  $\circledast$

- From  $\oplus$ :

- $$\underbrace{x \oplus_{\alpha,\beta} x \oplus_{\alpha,\beta} \cdots \oplus_{\alpha,\beta} x}_n = E_{\alpha,\beta}(E_{\alpha,\beta}^{-1}(x)n)$$
- Generalized to  $z = x \oslash_{\alpha,\beta} y = E_{\alpha,\beta}(E_{\alpha,\beta}^{-1}(x)y)$ .
- $x \oslash_{\alpha,\beta} -1$  is the inverse of  $x$  w.r.t.  $\oplus$

I have chosen asymmetrical symbols, to express they are non-commutative.

# Outline

- 1 The question
- 2 Characterize the exponential
- 3 LCA Groups
- 4 Transforms**
- 5 More groups
- 6 Time scales or measure chains
- 7 Summary
- 8 Future work

# Mittag-Leffler LCA: Pontryagin duality

- We are going to apply Pontryagin duality [11].
- $\otimes_{\alpha, \beta}$ : Locally  $\sigma$ -compact, Abelian, Hausdorff ( $\mathbb{R}$  is, so this is too).
- We need a Haar measure: non-negative linear functional, Radon measure (we use Riesz representation theorem), and invariant w.r.t. the action of the group. Unique up to a scalar.
- We find the multiplicative characters (continuous function to the unit circle which is an homomorphism):  $\chi_\alpha(x \otimes_\alpha y) = \chi_\alpha(x)\chi_\alpha(y)$ , but in the unit circle.

---

[11]. H. Loomis. *Introduction to abstract harmonic analysis*. New York: Dover Publications, 2011. ISBN: 9780486481234.

# Mittag-Leffler LCA: Pontryagin duality

- We are going to apply Pontryagin duality [12].
- $\circledast_{\alpha,\beta}$ : Locally  $\sigma$ -compact, Abelian, Hausdorff ( $\mathbb{R}$  is, so this is too).
- We can find a linear functional which is a Haar measure (invariant w.r.t. the action of the group).
- Given  $S$  a compact subset of the group:

$$\begin{aligned}\mu(S) &= \int_{S \circledast_{\alpha} a} \frac{E'_{\alpha}(x)}{E_{\alpha}(x)} dx, \text{ by compactity,} \\ &= \sum_n \int_{x_n \circledast_{\alpha} a}^{y_n \circledast_{\alpha} a} \frac{E'_{\alpha}(x)}{E_{\alpha}(x)} dx \\ &= \sum_n \ln(E_{\alpha}(x_n \circledast_{\alpha} a)) - \ln(E_{\alpha}(y_n \circledast_{\alpha} a)) \\ &= \sum_n \ln(E_{\alpha}(x_n)E_{\alpha}(a)) - \ln(E_{\alpha}(y_n)E_{\alpha}(a)) \\ &= \sum_n \ln(E_{\alpha}(x_n)) - \ln(E_{\alpha}(y_n)) = \int_S \frac{E'_{\alpha}(x)}{E_{\alpha}(x)} dx.\end{aligned}$$

---

[12] L. H. Loomis. *Introduction to abstract harmonic analysis*. New York: Dover Publications, 2011. ISBN: 9780486481234.

# Mittag-Leffler LCA: Pontryagin duality

- We find the multiplicative characters (continuous function to the unit circle which is an homomorphism):  $\chi_\alpha(x \circledast_\alpha y) = \chi_\alpha(x)\chi_\alpha(y)$ .

$$\chi_{\alpha,\gamma}(x) = e^{\beta 2\pi i \ln(E_{\alpha,\gamma}(x))} = E_{\alpha,\gamma}(x)^{\beta 2\pi i}. \quad (3)$$

- Use Pontryagin duality to define a transform (can be inverted by duality) invariant w.r.t the group:

$$\hat{f}(\beta) = \int_G f(x) \overline{\chi_{\alpha,\gamma}(x)} d\mu(x) = \int_{-\infty}^{\infty} f(x) E_{\alpha,\gamma}(x)^{-\beta 2\pi i - 1} E'_{\alpha,\gamma}(x) dx. \quad (4)$$

# Mittag-Leffler LCA: Pontryagin duality

- The same can be done for  $\oplus_{\alpha,\gamma}$ :

$$\chi_{\alpha,\gamma}(x) = e^{\beta 2\pi i E_{\alpha,\gamma}^{-1}(x)}. \quad (5)$$

$$\mu(S) = \int_S E_{\alpha,\gamma}^{-1'}(x) dx = \int_S \frac{1}{E'_{\alpha,\gamma}(E_{\alpha,\gamma}^{-1}(x))} dx. \quad (6)$$

- And for  $\odot_{\alpha,\gamma}$ , introduced later.

# Mittag-Leffler LCA groups

- For  $\alpha = 1$ ,  $E_1(x) = e^x$ , and we recover the regular Fourier transform.

$$\hat{f}(\beta) = \int_G f(x) \frac{e^{-x\beta 2\pi i} e^x}{e^x} dx = \int_{-\infty}^{\infty} f(x) e^{-x\beta 2\pi i} dx. \quad (7)$$

- For  $\alpha = 0$ ,  $E_0(x) = \frac{1}{1-x}$ , we obtain a variant of the Mellin transform

$$\hat{f}(\beta) = \int_{G>1} f(x) \frac{(1-x)^{1+\beta 2\pi i}}{(1-x)^2} dx - \int_{G<1} f(x) \frac{(1-x)^{1+\beta 2\pi i}}{(1-x)^2} dx \quad (8)$$

$$= \int_{-\infty}^1 f(x) (1-x)^{\beta 2\pi i - 1} dx - \int_1^{\infty} f(x) (1-x)^{\beta 2\pi i - 1} dx. \quad (9)$$

- Even closer to Mellin transform for  $\alpha = 0$ ,  $E_0(-x) = \frac{1}{1+x}$ .

# Pontryagin duality

- We can define the convolution of two functions in our space as:

$$(f *_{\alpha, \gamma} g)(x) = \int_G f(y) g((y \mapsto_{\alpha, \gamma} -1) \circledast_{\alpha, \gamma} x) d\mu_{\alpha, \gamma}(y). \quad (10)$$

From Pontryagin:

$$\widehat{(f *_{\alpha, \gamma} g)}(x) = \widehat{f}(x) \widehat{g}(x). \quad (11)$$

# Outline

- 1 The question
- 2 Characterize the exponential
- 3 LCA Groups
- 4 Transforms
- 5 More groups**
- 6 Time scales or measure chains
- 7 Summary
- 8 Future work

# General group composition

- We can combine groups recursively to create new group operations:

$$a \circledast_{\alpha} \oplus_{\beta} b = E_{\alpha}^{-1}(E_{\alpha}(a) \oplus_{\beta} E_{\alpha}(b)) = E_{\alpha}^{-1}(E_{\beta}(E_{\beta}^{-1}(E_{\alpha}(a)) + E_{\beta}^{-1}(E_{\alpha}(b)))).$$

- Right to left (so the domains fit):

$$\circledast^{\alpha,\beta} \underbrace{(\oplus^{\alpha,\beta} \circledast^{\alpha,\beta})}_{\text{correct}} = \circledast^{\alpha,\beta} * = \circledast^{\alpha,\beta} \underbrace{(\circledast^{\alpha,\beta} \oplus^{\alpha,\beta})}_{\text{incorrect}} \circledast^{\alpha,\beta} = + \circledast^{\alpha,\beta} \neq \circledast^{\alpha,\beta}$$

- Some combinations:

$$\circledast_{\alpha,\beta} \oplus_{\alpha,\beta} = \circledast_{1,1} = +,$$

$$\oplus_{\alpha,\beta} \circledast_{\alpha,\beta} = \oplus_{1,1} = *,$$

$$\circledast_{\alpha,\beta} * = \circledast_{\alpha,\beta},$$

$$\oplus_{\alpha,\beta} + = \oplus_{\alpha,\beta},$$

$$\oplus_{\alpha,\beta} * = \odot_{\alpha,\beta}.$$

- Where  $\odot_{\alpha,\beta}$  is a new commutative group operation:

$$z = x \odot_{\alpha,\beta} y = x \oslash_{\alpha,\beta} E_{\alpha,\beta}^{-1}(y) = E_{\alpha,\beta}(E_{\alpha,\beta}^{-1}(x)E_{\alpha,\beta}^{-1}(y)), \text{ (it has its own invariant transform also).}$$

# Outline

- 1 The question
- 2 Characterize the exponential
- 3 LCA Groups
- 4 Transforms
- 5 More groups
- 6 Time scales or measure chains**
- 7 Summary
- 8 Future work

# Analysis on measure chains (or time scales)

- Combine discrete and continuous analysis on  $\mathbb{T}$ , the domain of the time scale [13]. They define  $\oplus$  by:  $e_\alpha(t)e_\beta(t) = e_{\alpha \oplus \mu \beta}(t)$ ,  $\ominus t$  is the inverse.
  - Standard calculus  $\mathbb{T} = \mathbb{R}$ :

$$\sigma(t) = t, \text{ (forward jump)}$$

$$\mu(t) = 0, \text{ (graininess)}$$

$$\ominus t = -t, \text{ (inverse of } \oplus \text{),}$$

$$e_\alpha(t) = e^{\alpha t}, \text{ (exponential)}$$

$$\int_0^\infty f(t) \Delta t = \int_0^\infty f(t) dt.$$

- Nabla calculus  $\mathbb{T} = \mathbb{Z}$ :

$$\sigma(t) = t + 1, \text{ (forward jump)}$$

$$\mu(t) = 1, \text{ (graininess)}$$

$$\ominus t = -\frac{t}{1-t}, \text{ (inverse of } \oplus \text{)}$$

$$e_\alpha(t) = (1 - \alpha)^t, \text{ (exponential)}$$

$$\int_0^\infty f(t) \Delta t = \sum_{t=0}^\infty f(t).$$

For Nabla  $a \oplus^\mu b = a + b + \mu ab$ .

[13] M. Bohner and A. Peterson. *Dynamic equations on time scales: an introduction with applications*. Boston, MA: Birkhäuser, 2001.

## Analysis on measure chains (or time scales)

- Laplace transform for  $f(x)$ , in timescales with  $f(x) \neq 0$  when  $x > 1$

$$\mathcal{L}\{x\}(s) = \int_0^{\infty} x(t) e_{\ominus s}^{\sigma}(t, 0) \Delta t. \quad (12)$$

- For nabla calculus:

$$\mathcal{L}\{x\}(s) = \sum_{t=1}^{\infty} (1-s)^{t-1} x(t). \quad (13)$$

- The Fourier transform associated with  $E_0(x)$  is

$$\hat{f}(\beta) = \int_{-\infty}^1 f(x)(1-x)^{\beta 2\pi i - 1} dx - \int_1^{\infty} f(x)(1-x)^{\beta 2\pi i - 1} dx.$$

- It can be rewritten as (if  $s = 2\pi\beta i$ ,  $x > 1$  and  $1 = id$ )

$$\begin{aligned} \mathcal{L}\{f\}(s) &= \int_{id}^{\infty} f(x) E_{\alpha, \beta}(x)^{-s-1} E'_{\alpha, \beta}(x) dx \\ &= \int_{id}^{\infty} f(x) E_{\alpha, \beta}(inv(x) \uparrow (s+1)) E'_{\alpha, \beta}(x) dx. \end{aligned}$$

- Note that  $E_{\alpha, \beta}(x)^{-t} = E_{\alpha, \beta}(x \uparrow -t) = E_{\alpha, \beta}(inv(x) \uparrow t)$ .

# Analysis on measure chains (or time scales)

- Nabla calculus: the exponentials are the hypercube versions  $E_0(x)$ .
- Regular calculus: the exponentials are the regular ones  $E_1(x) = e^x$ : cross-polytypes
- What about  $E_{\frac{1}{2}}(x)$ ? (hyperspheres), discrete version? cuboctahedron?

# Outline

- 1 The question
- 2 Characterize the exponential
- 3 LCA Groups
- 4 Transforms
- 5 More groups
- 6 Time scales or measure chains
- 7 Summary**
- 8 Future work

# Contributions

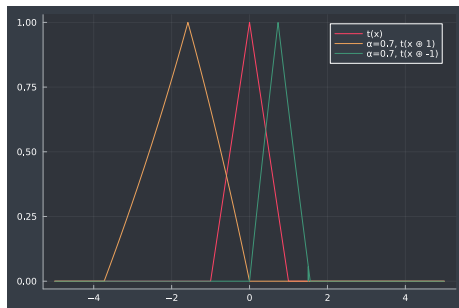
- Exponentials/Mittag-Leffler functions as generation functions of volumes.
- Supersum/submultiplications, many other interesting groups.
- New family of transforms, can be shifted from Fourier to Mellin.
- New way to look at transforms in time scales (and Fourier transform there), related both.

# Outline

- 1 The question
- 2 Characterize the exponential
- 3 LCA Groups
- 4 Transforms
- 5 More groups
- 6 Time scales or measure chains
- 7 Summary
- 8 Future work**

# Future ideas/Future work

- Study how the transforms change with translation/dilation in intermediate  $0 < \alpha < 1$ .



# Future ideas/Future work

- Semigroups  $p < 1$  i.e.  $\alpha > 1$ .
- Related: monomials  $x \mapsto_{\alpha,\beta} n$  (Study them in depth):
  - ▶  $x \mapsto^{0,1} n$  similar to  $x^n$ .
  - ▶  $x \mapsto^{1,1} n = xn$ .
  - ▶  $x \otimes_{2,1} n$ : Chebyshev polynomials but  $x \geq 1$  (with  $+$  and  $*$ ): can be half-composed (index measures composition) ( $x \otimes^{2,1}$  are also Chebyshev polynomials but  $x \leq -1$ ).
  - ▶ No other monomials are polynomials (with  $+$  and  $*$ ) unfortunately.
- Polynomials with combined groups?
- When  $m \geq 2$ , maybe  $E_{m,\beta}(x^m)$ , for example for Chebyshev.

## Future ideas/Future work

- Tetrahedral exponential function from outer radius  $R$  (hypersimplices):

$$O_n(R) = \frac{((n+1)^{n+1} n^{-n})^{\frac{1}{2}} R^n}{n!}. \quad (14)$$

$$f(R) = \sum_{n=0}^{\infty} O_n(R) = 1 + \sum_{n=1}^{\infty} \frac{(n+1)^{\frac{1}{2}} (1+1/n)^{\frac{n}{2}} R^n}{n!}. \quad (15)$$

- Tetrahedral exponential function from edge  $x$ : polyexponential functions (hypersimplices):

$$\sum_{n=0}^{\infty} V_n(x) = \sum_{n=0}^{\infty} \frac{(n+1)^{\frac{1}{2}} x^n}{2^{\frac{n}{2}} n!} = e_{\frac{1}{2}}(x/\sqrt{2}, 1). \quad (16)$$

(Hardy wrote: “All the functions which have been considered so far are in many ways analogous to the ordinary exponential function”).

- Geometry of exponentials: what does it really mean? Combinatorics?
- Study in depth the erfc case (hyperspheres): they must be special (noise is symmetric?).

# Future ideas/Future work

- Relate  $\otimes_{\alpha,\beta}$  and  $\otimes^{\alpha,\beta}$ , from  $E_{\alpha,\beta}(x)$  and  $E_{\alpha,\beta}(-x)$ .
- Relate  $\otimes_{\alpha,\beta}$  and  $\oplus_{\alpha,\beta}$ .
- Is there a  $\mathbb{T}$  for each  $\alpha, \beta$ ?
- $\odot_{\alpha,\beta}$  and  $\oplus_{\alpha,\beta}$  distribute? form a field? can we do vectors, etc?

# Future ideas/Future work

- Properties of the transforms [14] Plancherel, relate to  $Z$  transform. . .
- If we have a quadratic/sublinear transfer function: Fourier analysis.
- Discretization of the transforms (on exponential numbers  $E_{\alpha,\beta}(n)$ )).
- $N$ -dimensional transforms. Hankel like transforms (spherical  $n$ -dimensional).
- Are Mittag-Leffler functions related to rotations in some Minkowski (Banach) space?

---

[14] Semen B Yakubovich and Yuri Luchko. *The hypergeometric approach to integral transforms and convolutions*. Vol. 287. Springer Science & Business Media, 2012.

# References I

- [1] Gorka Guardiola Muzquiz. “Mittag–Leffler functions, lpn-balls and geometric pairings”. In: *Integral Transforms and Special Functions* 28.3 (2017), pp. 195–211. DOI: 10.1080/10652469.2016.1267730.
- [2] H. J. Haubold, A. M. Mathai, and R. K. Saxena. “Mittag-Leffler functions and their applications”. In: *J. Appl. Math.* 2011 (2011), pp. 1–51. DOI: 10.1155/2011/298628.
- [3] A. Erdélyi et al. (*Based on notes left by Harry Bateman*) *Higher transcendental functions*. Vol. III. Melbourne: Robert E. Krieger Publishing Co. Inc., 1981.
- [4] Rudolf Gorenflo et al. *Mittag-Leffler functions, related topics and applications*. Springer, 2020.
- [5] T. R. Prabhakar. “A singular integral equation with a generalized Mittag-Leffler function in the Kernel”. In: *Yokohama Math. J.* 19.1 (1971), pp. 7–15.
- [6] H. Pollard. “The completely monotonic character of the Mittag-Leffler function”. In: *Bull. Amer. Math. Soc.* 54.12 (1948), pp. 1115–1116.

# References II

- [7] R. K. Saxena and M. Saigo. “Certain Properties of Fractional Calculus Operators Associated with Generalized Mittag-Leffler Function”. In: *Fract. Calc. Appl. Anal.* 8.2 (2005), pp. 141–154. ISSN: 1311-0454.
- [8] X. Wang. “Volumes of generalized unit balls”. In: *Math. Mag.* 78.5 (2005), pp. 390–395.
- [9] K. S. Miller. “A note on the complete monotonicity of the generalized Mittag-Leffler function”. In: *Real Analysis Exchange* 23.2 (1999), pp. 753–756.
- [10] L. H. Loomis. *Introduction to abstract harmonic analysis*. New York: Dover Publications, 2011. ISBN: 9780486481234.
- [11] M. Bohner and A. Peterson. *Dynamic equations on time scales: an introduction with applications*. Boston, MA: Birkhäuser, 2001.
- [12] Semen B Yakubovich and Yury Luchko. *The hypergeometric approach to integral transforms and convolutions*. Vol. 287. Springer Science & Business Media, 2012.